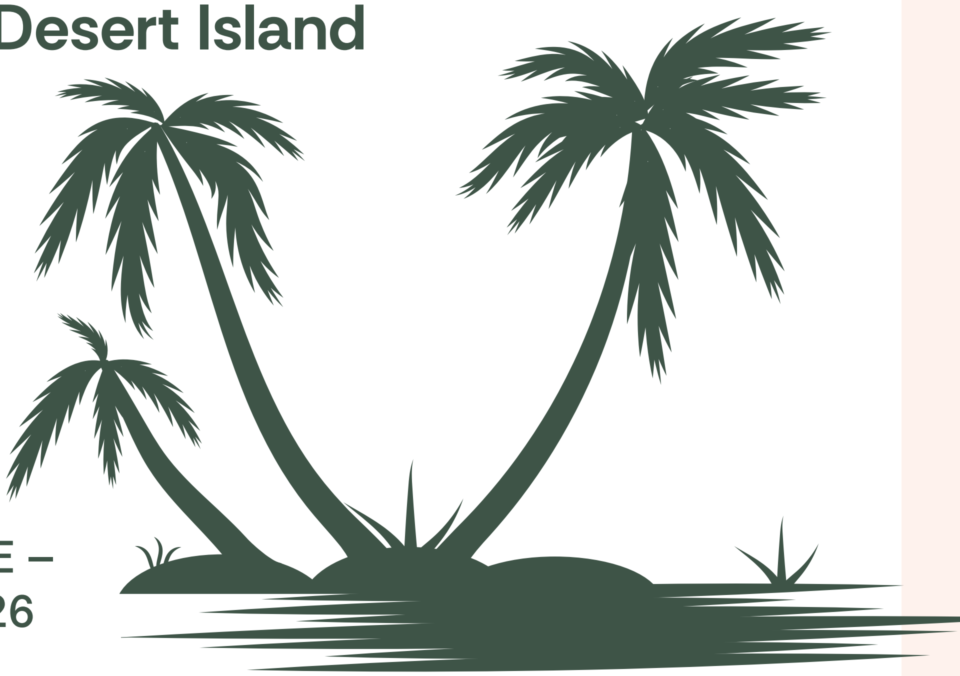
Mathematics for a Desert Island



Professor Paul Glaister CBE – AMiE President 2025 - 2026

Listening to the BBC's Desert Island Discs (DID) programme over many years, I have often thought about which eight music tracks I would choose to have with me if I was unfortunate to be cast away alone on a desert island. This would be somewhat of an impossible task as there is so much music that I enjoy listening to which I would not want to be without. As for a 'luxury item' I'd probably go for a Nespresso machine with a limitless supply of capsules, and my 'book of choice' would likely be related to cricket.

This got me thinking recently about an alternative proposition – what if all my choices had to be related to mathematics? For my 'book' I would 'try my luck' as other castaways have done and opt for the set of collected works - '*University Mathematical Texts*' published by Oliver & Boyd[1]. Although this collection comprises 38 titles, none of them are tomes, so at the very least I hope I will be allowed to have 20 or so of those that I already have. There is much in these works that I have either forgotten about or not studied in great depth, and I will have plenty of uninterrupted time to enjoy learning and 'catching up'.

I would also like to continue research in numerical analysis[2] and computational fluid dynamics[3], including work on systems of hyperbolic conservation laws[4], Euler equations[5], compressible fluid flow[6], St Venant/shallow water equations[7], shock waves[8], and real gases[9], which I have enjoyed doing and have made some progress on over the years[10].

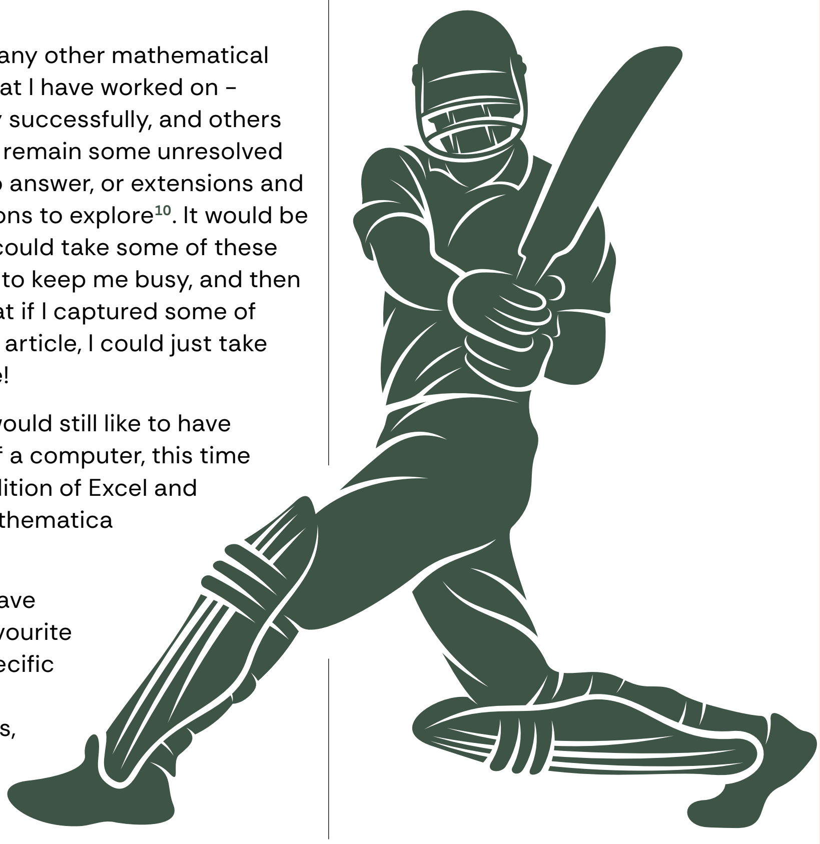
To be able to do this my luxury would need to be a high-performance computer with software capable of running Python code. The usual 'rule' on DID is that one cannot take a luxury item that has some practical use to aid my survival on the island, and I am confident that this would be highly unlikely in this case, although it would at least help me to retain my sanity!

There are many other mathematical problems that I have worked on - some wholly successfully, and others where there remain some unresolved questions to answer, or extensions and generalisations to explore[10]. It would be a bonus if I could take some of these with me too to keep me busy, and then I thought that if I captured some of these in this article, I could just take that with me!

Naturally, I would still like to have the luxury of a computer, this time with the addition of Excel and Wolfram Mathematica (or similar).

You will all have your own favourite areas, or specific topics in mathematics, problems that you enjoyed working on and solving, other problems whose solution has so far eluded you, or conjectures that you have yet to establish as a proven theorem or discovered a counterexample for.

What follows are 15 of the many problems I have enjoyed working on[10], most of which have been fully resolved and which you might like to explore for yourself. I have also captured some other problems in another compendium[11].



I encourage you all to find some time for reflection and put together your own list of mathematical problems, resources etc. you would like to have with you if you were cast away on a desert island.

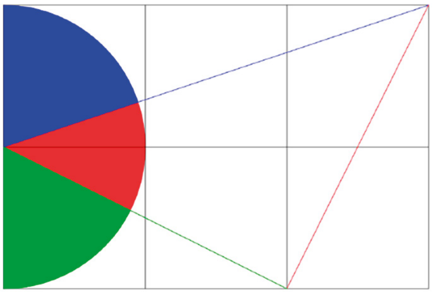
1. Find all possible natural number solutions m, n, p of $\frac{m}{n} = 10^{-p}(n - m)$ or $\frac{n}{m} = 10^{-p}(n - m)$.

   For example, a solution of the first equation is $(m, n, p) = (81,90,1)$, i.e. $\frac{81}{90} = 0{\cdot}1 \times (90 - 81)$, and a solution of the second equation is $(m, n, p) = (10201,101,2)$, i.e. $\frac{10201}{101} = 0{\cdot}01 \times (10201 - 101)$.

2. The well-known result:

   $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

   as illustrated in the figure below



   stimulates many questions.

   For example one can readily find other sequences of natural numbers $a_1, a_2, \dots, a_n$ for which

   $\tan^{-1} a_1 + \tan^{-1} a_2 + \cdots \tan^{-1} a_n = m\pi$

   for natural numbers m, but what about the more specific problem of finding natural numbers $n \neq 3$ for which

   $\tan^{-1} 1 + \tan^{-1} 2 + \cdots + \tan^{-1} n = m\pi$

   for natural numbers m?

3. One can have much fun with proof involving finite and infinite series, particularly those incorporating the binomial coefficients $\binom{k}{r} = \frac{k!}{(k-r)!r!}$, $0 \leq r \leq k$, two examples of which are:

   $$\binom{n-1}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{4}\binom{n+1}{2} + \frac{1}{8}\binom{n+2}{3} + \cdots$$
   $$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

   and

   $$\frac{1}{1}\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \cdots + (-1)^n \frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}.$$

   I was once asked if the first of these had any practical use, to which my reply was 'I have no idea, but in any case, who cares!' But to my surprise some years later I discovered that one of my articles that included this had been cited in work on 'delay optimisation and performance analysis of mobile edge computing for autonomous aerial vehicles enabled for the internet of ground vehicles' and also cited in work on 'Catalan triangles and Finucan's hidden folders'!

   Most weeks I receive notifications of new citations of my published works[10]; it is heartening to learn that some of my mathematical endeavours have been of value to someone else.

4. Infinite series with Fibonacci numbers F_i defined by $F_{i+2} = F_{i+1} + F_i$, $i = 1, 2, \dots$; $F_1 = F_2 = 1$, provide some interesting challenges. For example, determine each of the following sums:

   $\frac{1}{2} \times F_1 + \frac{1}{4} \times F_2 + \frac{1}{8} \times F_3 + \cdots$

   $\frac{1}{2} \times F_1 \times 1 + \frac{1}{4} \times F_2 \times 2 + \frac{1}{8} \times F_3 \times 3 + \cdots$

   $F_1 - \frac{1}{2} \times F_2 + \frac{1}{4} \times F_3 - \cdots$

   $F_1 \times 1 - \frac{1}{2} \times F_2 \times 2 + \frac{1}{4} \times F_3 \times 3 - \cdots$

   $F_2 - \frac{1}{2} \times F_3 + \frac{1}{4} \times F_4 - \cdots$

   $F_2 \times 1 - \frac{1}{2} \times F_3 \times 2 + \frac{1}{4} \times F_4 \times 3 - \cdots$

5. A further type of problem with infinite series is to determine the values of the parameter, t, for which the sum of each of the following are natural numbers:

   $1 + t + t^2 + \cdots$

   $1 + 2t + 3t^2 + \cdots$

   $1^2 t + 2^2 t^2 + 3^2 t^3 + \cdots$

   $F_1 t + F_2 t^2 + F_3 t^3 + \cdots$

6. Conic sections are curves formed by the intersection of a plane with a cone. The four main types of conic section are the circle, ellipse, parabola, and hyperbola, depending on the angle of the intersecting plane. These curves have applications in many fields.

One traditional way of constructing the latter 3 curves is to consider the loci of points whose distance from a fixed point (a focus) to a fixed line (a directrix) are in a constant ratio (the eccentricity).

If one considers the locus of points that are equidistant from two concentric circles this obviously results in a third concentric circle between the circles. But what happens when the two circles are not concentric, i.e. when they do not have the same centre?

7. Given natural numbers a, k, b, where $a < k < b$, for what triples (a, k, b) will $y = kx$ bisect the two straight lines $y = ax, y = bx$? One solution is $(a, k, b) = (1, 2, 7)$, which is one of a family of solutions with $k = 2a$, another one of which is $(a, k, b) = (7, 14, 1393)$.

However, there are solutions which are not members of this family, for example

$(a, k, b) = (7, 12, 41)$, where $a = 7$ as in the previous solution. What family of solutions is this one a member of, and are there other solution families?

8. Determining inverses, eigenvalues, or even determinants of special matrices also provide opportunities for exploration. For example, what are the determinants of the following square matrices?

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

The third one is probably the more difficult challenge of these, although all three can be tackled through pattern spotting, making a conjecture and then proving it.

9. A similar problem involving pattern spotting, making a conjecture and proving it is to determine the n^{th} derivative of

$f(x) = \frac{x}{1+x^2}$ at $x = 1$, i.e. $f^{(n)}(1)$.

I first looked at this problem about 30 years ago when investigating the nature of stationary points of this function. At the time Wolfram Mathematica did not produce the general result, and 'exceeded maximum computation time' for moderately large values of n. Of course this is no longer the case. [One approach to solving this problem 'by hand' is to rewrite as $(1 + x^2)f(x) = x$ and differentiate n times with respect to x using Leibniz's rule for differentiating products and involving binomial coefficients again.]

10. A standard problem in particle dynamics is to consider the motion of an object which is projected vertically upwards in a uniform gravitational field and where gravity is assumed to be the only force acting. With initial speed of projection U, constant acceleration due to gravity g, the particle takes a time $T_u = \frac{U}{g}$ to reach the highest point. It also takes the same time $T_d = \frac{U}{g}$ to travel back down to the point of projection, returning to that point with speed V which is the same as the initial speed, i.e. $V = U$.
This means that the total time of flight

$$T = \frac{2U}{g} = \frac{U+V}{g} = \frac{\text{initial speed} + \text{final speed}}{g}.$$

Variations on this problem include considering the effect of additional forces, such as that due to air resistance of the form $f(v)$ where $v(t)$ represents the velocity of the object at any time *t* of the flight, with typical models being of the form $f(v) = \alpha v + \beta v^2$, for constants α, β, where at least one of α and β is non-zero.

For general $f(v)$, or these more specific forms, one can determine the corresponding expressions for T_u, T_d, T, V as functions of U. Comparisons can be made between T_d and T_u and between V and U in the general case, and for the specific models above.

One interesting aspect is to consider for what law of resistance $f(v)$ is the time of flight T also

$$\frac{U+V}{g} = \frac{initial\ speed+final\ speed}{g}$$

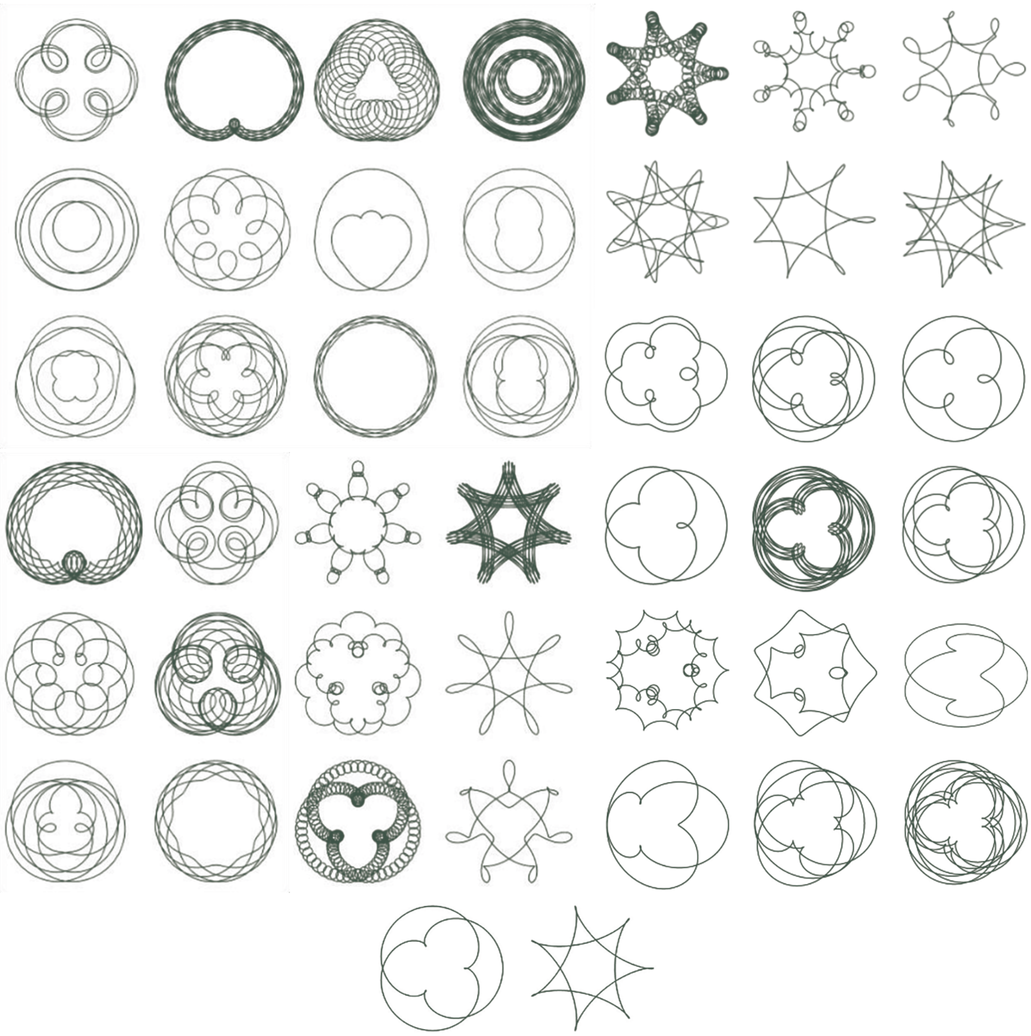
as in the case with no resistance with $f(v) \equiv 0$.

11. Another standard problem concerning particle motion in a uniform gravitational field (with or without air resistance) is to determine the angle of projection such that the object attains its maximum range on a horizontal plane. But what about the actual *distance* travelled by the particle, i.e. for what angle of projection would the *path length* attain its *maximum* value?

12. Have you ever knocked over an empty tumbler (hopefully plastic!), or a full one? Its stability will depend on the location of the centre of mass of the whole system – tumbler plus liquid. Consider determining the location of the centre of mass of such a system as the tumbler is gradually filled with liquid, noting how it varies. There is a special property for the location of the centre of mass when it is at its *lowest* point – what is this property, and can you prove it?

13. You will likely be aware that it takes longer for the 'last half' of a bath to empty compared with the 'first half', which first captured my interest decades ago when I was bathing our two children when they were toddlers and they left me to sort out the toys that had been left in the bath! A simple model using physics and calculus predicted that it would take $\sqrt{2} + 1$ times longer for the last half to drain compared with the first half, which I then tested and found to my surprise was quite an accurate estimate.

14. What dynamical system results in the following trajectories in 2 dimensions?



15. For the last problem, consider separating the first $2n$ natural numbers, where $n \geq 1$, into two sets $A = \{1, 2, ..., n\}$ and $B = \{n+1, n+2, ..., 2n\}$, and form a sequence, or '*matching*', of n pairs of the form (a, b) where a is chosen from the set A, and b is chosen from the set B.

   i. Determine how many such matchings there are as a function of n.

   For example, with $n = 3$, there are 6 possible matchings:

a	1	2	3
b	4	5	6

a	1	2	3
b	4	6	5

a	1	2	3
b	5	4	6

a	1	2	3
b	5	6	4

a	1	2	3
b	6	4	5

a	1	2	3
b	6	5	4

   ii. For a given n, is there a matching where for every pair (a, b), a and b are 'coprime' (or 'relatively prime', i.e. they share no common factors other than 1 so that their greatest common divisor 1? We call this a coprime matching.

   In the example above we see that only for the last matching:

a	1	2	3
b	6	5	4

   is it a coprime matching.

   Is there at least one coprime matching for every n?

   iii. For the first few values of n the number of coprime matchings $f(n)$ is as follows:

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	1	1	2	6	12	44	132	504	2016

   What is the value of $f(11)$? Can you find determine a formula for $f(n)$?

   iv. One can also ask – for what values of n is the 'reverse matching':

a	1	2	...	$n-1$	n
b	$2n$	$2n-1$	...	$n+2$	$n+1$

   a coprime one, and what other mathematical problems are related to this, and why?

As I said at the start of my Presidency, I have celebrated the beauty, power and joy of mathematics throughout the last fifty years since I started studying A level Mathematics. I hope all of you are able to celebrate these wonders of mathematics with one another as we come together as a community within AMiE.

In keeping with the quote by the famous Hungarian mathematician Alfréd Rényi[12]

"A mathematician is a machine for turning coffee into theorems."

I will definitely be needing my Nespresso machine and capsules on my desert island!

Endnotes

1 Oliver and Boyd University Mathematical Texts
2 Numerical analysis
3 Computational fluid dynamics
4 Hyperbolic partial differential equations
5 Euler equations
6 Compressible flow
7 Shallow water equations
8 Shock waves
9 Real gases
10 Items where Author is "Glaister, Professor P" - CentAUR
11 A mathematicians miscellany and an apology Paul Glaister
12 Coffee into Theorems

AMiE President

Prof Paul Glaister CBE

Celebrating the beauty, power and joy of Mathematics

Throughout Paul's presidency he has explored 'Celebrating the beauty, power and joy of Mathematics'.

Paul has two previous president reports, issued from earlier in his presidency. These can be accessed within the Member's Area or by using the QR codes.

You can read them here:

From Coasting to Commanding

Curriculum and assessment review and reform: evolutions or revolution?

Prof Sarah Hart will be AMiE President from April 2026-27, following on from Prof Paul Glaister CBE.

Sarah is a mathematician and author. She is Professor Emerita of Mathematics and Fellow of Birkbeck College (University of London), and has recently served a four-year term as Professor of Geometry at Gresham College, the first woman to hold this chair since its creation in 1597.

Sarah studied at Oxford and Manchester, gaining her PhD in 2000. Postdoctoral research and teaching followed, including a prestigious Engineering and Physical Sciences Research Fellowship, before she was appointed to a lectureship at Birkbeck in 2004. Her research is largely in group theory, which is the main tool used by mathematicians to understand symmetry. She became Professor of Mathematics at Birkbeck in 2013, and served as Assistant Dean for Retention and Widening Participation from 2012-2016 and Head of the Department of Economics, Mathematics and Statistics from 2016-2019. She also served a three-year term as President of the British Society for the History of Mathematics from 2021-23.

As a hugely popular lecturer with over 20 years teaching experience, she is a regular speaker at international events, and has written for publications including the *New York Times* and *New Scientist*. Sarah is particularly interested in the cultural, historical and creative impact of mathematics, and the links between mathematics and the arts. Her first book, *Once Upon a Prime: The Wondrous Connections Between Mathematics and Literature*, published in 2023, was a New York Times Book review Editor's Choice, and won the Mathematical Association of America's Euler Book Prize in 2024.

On being invited to become president, Sarah said "I'm honoured to have been chosen to serve as President. I'm looking forward to working with this brilliant organisation to support mathematics and mathematics educators in any way I can. As I'm especially interested in the links between mathematics and other creative subjects, during my Presidency I would love to explore ways to increase opportunities for cross-disciplinary connections with the arts and humanities."